PARTICULAR LOADS OF THE ELASTIC HALF-SPACE

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Abstract: *Since the dimensions of the contact area are small compared to the curvature radii of the two adjacent surfaces, evaluated in the initial contact area, it is often considered that these radii are infinite. In this hypothesis, the bodies in contact can be assimilated through elastic half-spaces. Elastic half-space can be loaded by applying loads on its boundary plane which, in the simplest situations, can be concentrated or uniformly distributed along a straight line. The principle of superimposition allows to appreciate the effects of continuous load distributions, of interest, applied to regions of the boundary plan. In the paper, starting from a general case of normal load distribution on an infinitely long strip, the expressions of the normal displacements w are established.*

Keywords: *contact mechanics, elastic half-space, normal displacements, pressure distribution*

1. Introduction

The bodies in contact can be assimilated through elastic half-spaces. By elastic half-space is meant that part of the space limited by a plane, which is filled with an elastic material of known parameters ν (Poisson`s ratio), E (Young`s modulus) and G shear modulus).

Elastic half-space can be loaded by loads applied on its boundary plane. These forces can be concentrated or uniformly distributed along a straight line. The problems of determining the displacements and stresses produced in the half-space by these simple loads are called fundamental problems of the elastic half-space.

The case of loading the elastic half-space with a concentrated force, normal to the boundary plane, is called Boussinesq's problem. Loading the elastic half-space with a concentrated force contained in the boundary plane leads to Cerruti's problem. The combined Boussinesq-Cerruti problem arises when the half-space is loaded with an oblique concentrated force, applied at a point on the boundary of the half-space. If the half-space is loaded with a normal load, uniformly distributed along a line contained in the boundary plane, Flamant's problem appears. The same category of fundamental problems includes the principle of superimposition on the elastic half-space, which allows the generalization of the solutions to the above problems in the case of continuous load distributions on a certain region of the boundary plane.

2. Distributes loads on infinitely long strips, constant width

Let it be a strip of constant width *2b*, extended to infinity in the boundary plane of the elastic half-space, having *y* as the axis of symmetry and the *x*-axis perpendicular to *y* at an arbitrary point of its own. In a cross section on the strip, the load distribution is independent of y, $\mathbf{p} = \mathbf{p}(x) = p_x \mathbf{i} + p_y \mathbf{j} + p_z \mathbf{k}$. The displacements acquired by the points of the boundary plane in the normal direction to the half-space under the action of normal loads, $p_z = p(x)$, tangential tractions p_x and p_y being considered negligible or zero, will be determined, Ref. [1].

Following the methodology developed by Johnson for point contacts, Ref. [2], a general case will be addressed, in which the normal load distribution is given by the equation:

$$p(x) = p_0 \left(1 - \frac{x^2}{b^2}\right)^n \tag{1}$$

where p_0 represents the pressure value on the symmetry axis of the strip, and n is an exponent that can take various values. Thus, if $n = 0$, $p(x) = p_0$, and the loading is uniform on the strip of width 2b.

The case $n = 1/2$ represents a loading of semi-elliptical profile, and the case $n = -1/2$ is of interest for flat surface contact.

The loading is characterized by the load q distributed on the strip's length unit.

The equilibrium equation of the half-space takes the form:

$$q = \int_{-b}^{+b} p(x)\, dx = p_0 \int_{-b}^{+b} \left(1 - \frac{x^2}{b^2}\right)^n dx \tag{2}$$

The principle of superposition for the solution of Flamant's problem leads to the following expression of the normal displacement:

$$w(x,0) =$$
$$w = \frac{1}{2\pi G} \int_{-b}^{+b} p(x') \left[\frac{z^2}{r'^2} - 2(1-\nu)\ln\frac{r'}{r_0}\right]_{z=0} dx' =$$
$$-\frac{1-\nu}{\pi G} \int_{-b}^{+b} p(x') \ln\frac{|x-x'|}{r_0}\, dx' \tag{3}$$

If the pressure has the form [1] in dimensionless coordinates $\bar{x}' = x'/b$ and $\bar{x} = x/b$, equation [3] has the expression:

$$w(\bar{x}) =$$
$$-\frac{1-\nu}{2\pi G} p_0 b \int_{-1}^{+1} (1-\bar{x}'^2)^n \ln(\bar{x} - \bar{x}')^2 d\bar{x}' -$$
$$-\frac{1-\nu}{\pi G} q \ln\frac{b}{r_0}. \tag{4}$$

Uniform pressure distribution $n = 0$

Equation [2] leads to the following minimum pressure value:

$$p_0 = \frac{q}{2b} \tag{5}$$

so that expression [4] of displacement w becomes:

$$w(\bar{x}) =$$
$$-\frac{1-\nu}{4\pi G} q \int_{-1}^{+1} \ln(\bar{x} - \bar{x}')^2\, d\bar{x}' - \frac{1-\nu}{\pi G} q \ln\frac{b}{r_0} \tag{6}$$

The substitution $(\bar{x} - \bar{x}')^2 = t$, leads to the primitive $-(\bar{x} - \bar{x}')\left[\ln(\bar{x} - \bar{x}')^2 - 2\right]$ which, between the limits of integration -1 and $+1$, provides the following expression of the displacement w:

$$w(\bar{x}) =$$
$$= \frac{1-\nu}{4\pi G} q \left[4 - (1-\bar{x})\ln(1-\bar{x})^2 - \right.$$
$$\left. -(1+\bar{x})\ln(1+\bar{x})^2 - 4\ln\frac{b}{r_0}\right]. \tag{7}$$

On the symmetry axis of the loading strip, $\bar{x} = 0$, the displacement $w(\bar{x})$ becomes:

$$w(0) = \frac{1-\nu}{\pi G} q \left(1 - \ln\frac{b}{r_0}\right) \tag{8}$$

and at the strip's edges it has the expression:

$$w(\pm 1) = \frac{1-\nu}{\pi G} q \left(1 - \ln\frac{2b}{r_0}\right) \tag{9}$$

The central displacement of the half-space in relation to the contact strip edges:

$$\Delta w = w(0) - w(\pm 1) = \frac{1-\nu}{\pi G} q \ln 2 \qquad (10)$$

Pressure distribution with $n = -1/2$

The case $n = -1/2$ represents a load with a minimum pressure p_0 on the symmetry axis of the strip and tending to infinity on its edges. The minimum pressure value p_0 results from the specification of equation [2]:

$$p_0 = \frac{q}{\pi b} \qquad (11)$$

Relation [4] leads to the following integral expression of the displacement *w*:

$$w(\bar{x}) = = -\frac{1-\nu}{2\pi G} p_0 b \int_{-1}^{+1} \frac{\ln\left(\bar{x} - \bar{x}'\right)^2}{\sqrt{1-\bar{x}'^2}} d\bar{x}' - \frac{1-\nu}{2\pi G} q \ln \frac{b}{r_0} \qquad (12)$$

To explain the displacement, we resort to the evaluation of the derivative $\partial w / \partial \bar{x}$ on the interval $[-1, +1]$. Since the integral involved in relation [12] is parametric with respect to $\bar{x}$, the displacement derivative is:

$$\frac{\partial w}{\partial \bar{x}} = \frac{1-\nu}{\pi G} p_0 b \int_{-1}^{+1} \frac{d\bar{x}'}{(\bar{x}' - \bar{x})\sqrt{1-\bar{x}'^2}} \qquad (13)$$

Applying the substitution $t = 1/(\bar{x}' - \bar{x})$ in relation [13] leads to the conclusion that the derivative is zero inside the loaded strip. As a result, on the strip of width *2b*, the displacement *w* is constant.

The constant displacement value w_0 on the loaded area is obtained by calculating the integral [12] at any point of the strip, be it $\bar{x} = 0$:

$$w(\bar{x}) = w(0) = w_0 = \frac{1-\nu}{G} q \ln \frac{2r_0}{b}, \quad \text{for } |\bar{x}| \le 1 \qquad (14)$$

Outside the loaded strip, $|\bar{x}| > 1$, the same substitution leads to the following expression for the slope of the deformed surface profile:

$$\frac{\partial w}{\partial \bar{x}} = = -\frac{2(1-\nu)}{\pi G} p_0 \frac{1}{\sqrt{\bar{x}^2 - 1}} \arcsin \frac{1}{\sqrt{2\bar{x}^2 - 1}} \qquad (15)$$

Equation (15) shows that, at the limits of the loaded zone, the slope of the deformed surface is infinite, the angle of the tangent with the horizontal being $\pi/2$, and at infinity the tangent is horizontal.

Johnson, Ref. [2], gives the following solution for moving *w* outside the charged zone:

$$w(\bar{x}) = = \delta_z - \frac{1-\nu}{G} p_0 b \ln\left(\bar{x} + \sqrt{\bar{x}^2 - 1}\right) \qquad (16)$$

where δ_z is an integration constant, unspecified. This can be determined from the condition of displacement continuity at the edge of the loaded strip, where formula [16] must provide its value given by relation [15]. After its identification, the following expression of the displacement *w*, outside the loading strip, results:

$$w(\bar{x}) = \frac{1-\nu}{\pi G} q \ln \frac{2r_0}{b(\bar{x} + \sqrt{\bar{x}^2 - 1})} \qquad (17)$$

Pressure distribution with $n = 1/2$ **(hertzian distribution)**

The case $n = 1/2$ represents a load with maximum pressure p_0 on the symmetry axis of the strip and zero on its edges, known as Hertzian pressure. The maximum pressure value results, also in this case, by the particularization of equation [2]:

$$p_0 = \frac{2q}{\pi b} \qquad (18)$$

The displacement's expression w on the boundary of the half-space is obtained by the particularization of equation [4]:

$$w(\overline{x}) = = -\frac{1-\nu}{2\pi\, G} p_0\, b \int_{-1}^{+1} \sqrt{1-\overline{x}'^2}\ \ln(\overline{x}-\overline{x}')^2\ d\overline{x}' - -\frac{1-\nu}{\pi\, G} q \ln\frac{b}{r_0} \tag{19}$$

Since the integrand in equation [19] does not admit an obvious explicit primitive, it is convenient to evaluate the displacement's derivative w:

$$\frac{\partial w}{\partial \overline{x}} = -\frac{1-\nu}{\pi G} p_0\ b \int_{-1}^{1} \frac{\sqrt{1-\overline{x}^2}}{\overline{x}-\overline{x}'}\ d\overline{x}' \tag{20}$$

In the case of the points on the loaded strip, $1-\overline{x}^2 \geq 0$, the following expression of the deformed surface slope inside the loaded strip is obtained, after integration:

$$\frac{\partial w}{\partial \overline{x}} = -\frac{1-\nu}{G} p_0\ b\ \overline{x} \tag{21}$$

The solution of differential equation [21] is immediate:

$$w(\overline{x}) = -\frac{1-\nu}{2G} p_0\ b\ \overline{x}^2 + C \tag{22}$$

where C is an integration constant that is determined from the condition that along the strip's symmetry axis, at $\overline{x} = 0$, $w = w(0)$; results $C = w(0)$ and

$$w(\overline{x}) = w(0) - \frac{1-\nu}{2G} p_0\ b\ \overline{x}^2 \tag{23}$$

Relation [23] shows that, inside the loaded strip, the profile of the deformed surface is parabolic along the x direction. To calculate the central displacement, relation [19] is used, in which $w(0)$, is substituted, obtaining:

$$w(0) = = -\frac{2(1-\nu)}{\pi\, G} p_0\ b \int_{0}^{1} \sqrt{1-\overline{x}'^2}\ \ln \overline{x}'\ d\overline{x}' - -\frac{1-\nu}{\pi\, G} q \ln\frac{b}{r_0} \tag{24}$$

Since the primitive of the above integral cannot be expressed with the help of a finite combination of elementary functions, this integral is calculated numerically, obtaining:

$$w(0) = \frac{1-\nu}{\pi\, G} q \left(1{,}1932 - \ln\frac{b}{r_0}\right) \tag{25}$$

At the edge of the loaded strip, $\overline{x} = 1$, the displacement in the z direction, $w(\pm 1)$, has the value:

$$w(\pm 1) = \frac{1-\nu}{\pi\, G} q \left(0{,}1932 - \ln\frac{b}{r_0}\right) \tag{26}$$

The central displacement of the half-space in relation to the edges of the contact strip is:

$$\Delta w = w(0) - w(\pm 1) = \frac{1-\nu}{\pi G} q \tag{27}$$

3. Conclusions

Other types of loads are approached in the specialized literature. Thus, Love treats uniform pressure distribution on circular and rectangular areas. Galin treats uniform pressure distribution inside an acute angle and on long and narrow rectangular strips. and Fabrikant solves the problem of asymmetric loading with normal load on a circular area, respectively with an arbitrary distribution of tangential load, on the same type of area. Sackfield and Hills study the effects of uniform or semi-ellipsoidal shear stress distributions, applied to a rectangular area. Svec and Gladwell calculate the deformations of the free surface of the elastic half-space, generated by a continuous pressure distribution described by a cubic polynomial, applied to a triangular area. Chowdhury addresses the problem of the elastic half-space subjected to a concentrated moment, normal to the free surface, while Shibuya considers the twisting moment applied to a circular crown contained within the boundary of the half-space. Maiti, Bela Das and Palit use Somigliana's method to find the displacements produced by normal and tangential tractions applied on the half-space boundary, and Prescott treats the problem of the deformations of this boundary in cylindrical and spherical coordinates.

4. References

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2. [Johnson, 1985] Johnson, K.L., Contact mechanics, Cambridge University Press, 1985.