ROLLING VERSUS SLIDING IN DYNAMICAL SYSTEMS PART II: ANALYTICAL SOLUTIONS OF THE DYNAMICAL SYSTEM

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Abstract: *In the second part of the paper, analytical solutions for the problem of the movement of a cylinder on an inclined plane are presented. The two possible situations are considered: sliding motion (2DOF system) and rolling motion (1 DOF system). The obtained results are presented comparatively. The results of the pure rolling test (kinematic and kineto-static) are also presented. It is found that when the sliding conditions are not met, the solutions of the system with 2DOF coincide with those of the system with 1DOF. In other words, the differential equations derived for the case of sliding are also applicable to the case of pure rolling.*

Keywords: *dry fruction, degree of freedom, nonlinear differential solution*

1. Introduction

In the first part of the paper there were presented the components of the reaction torsor for a point contact between two solid bodies [1], resulting from the components of relative motions from the concentrated contact. Next, a simple system that consists in a homogenous cylinder of revolution which moves downward an inclined plane is presented; in this system, both sliding and rolling may manifest. The solution of the motion is simulated using a dynamic simulation software and presented at the final part. There were found the time variation for the velocity of the centre of mass of the cylinder, and for the angular velocity, for three pairs of values of the coefficients of static friction μ_{st} and dynamic friction, μ_d, for each situatuion testing the presence of pure rolling or sliding.

In the second part of the paper there are presented the analytical solutions for the motion of the system in the two possible solutions: sliding, with 2DOF or rolling with 1DOF. It is expected that the mathematical model corresponding to sliding is more intricate.

The dynamic model of the problem is presented in Figure 1. The motion of the cylinder is a plane-parallel one and the study concerns the frontal median section of the cylinder.

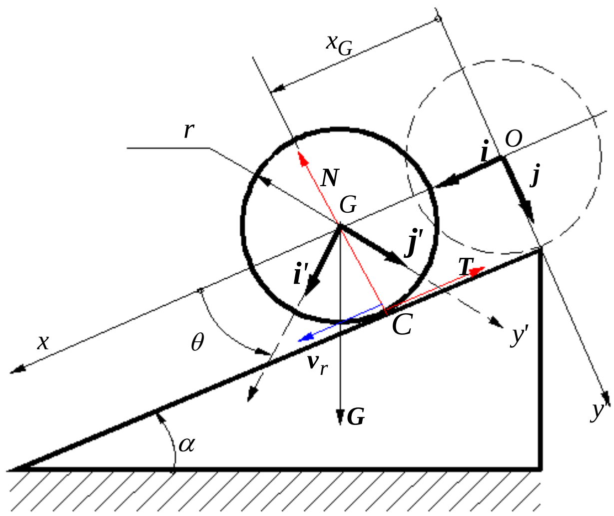


Figure 1. *Cylinder in downward motion on an inclined plane*

It is considered that the material of the cylinder is homogenous, of density $\rho = 7800\, kg/m^3$, and the geometry is defined by the radius $r = 0.12m$ and height $h = 0.2m$.

The inertial characteristics of the cylinder are the mass:

$$M = \pi r^2 h\rho \tag{1}$$

And the axial moment of inertia with respect to the axis of revolution [2]:

$$J_{Gz} = \frac{1}{2} M r^2 \tag{2}$$

In order to stipulate the position of the rigid body, two reference coordinate systems are used: the fixed frame $Oxyz$ and the mobile rame $Gx'y'z'$ that has the origin in the centre of mass G positioned at the distance x_G with respect to the origin of the immobile frame, and having the axes Ox' and Oy' at an angle of rotation θ with respect to the axes of the fixed system. The tilting angle of the inclined plane is α. In order to solve the problem, the theorem of momentum is applied under the particular form of the theorem of motion of the mass centre [2,3]:

$$M\boldsymbol{a}_G = \sum \boldsymbol{F} \tag{3}$$

And then, the moment of momentum theorem, with respect to the axis of rotation, is applied:

$$\boldsymbol{J}_G \varepsilon + \tilde{\omega} \boldsymbol{J}_G \omega = \sum \boldsymbol{M}_G \tag{4}$$

In the equations (3) and (4), the next notations are used: a_G is the acceleration of the centre of mass, F is the resultant of the forces acting upon the cylinder, for which characterisation, the projections on the axes of the immobile frame will be used; the forces are represented by:

- the weight G of the cylinder:

$$\boldsymbol{G} = \begin{bmatrix} Mg\sin\theta \\ Mg\cos\theta \\ 0 \end{bmatrix} \tag{5}$$

- the normal reaction N:

$$\boldsymbol{N} = \begin{bmatrix} 0 \\ -N \\ 0 \end{bmatrix} \tag{6}$$

- the friction force T:

$$\boldsymbol{T} = \begin{bmatrix} T \\ 0 \\ 0 \end{bmatrix} \tag{7}$$

The matrix of the moments of inertia J_G has a diagonal form since the axes of the frame $Ox'y'z'$ are principal inertia axes. Only the principal central moments of inertia occur in the matrix:

$$\boldsymbol{J}_G = \begin{bmatrix} J_{Gx} & 0 & 0 \\ 0 & J_{Gy} & 0 \\ 0 & 0 & J_{Gz} \end{bmatrix} \tag{8}$$

In relation (4), ε represents the angular acceleration:

$$\varepsilon = \begin{bmatrix} 0 \\ 0 \\ \varepsilon_z \end{bmatrix} \tag{9}$$

and ω is the angular velocity

$$\omega = \begin{bmatrix} 0 \\ 0 \\ \omega_z \end{bmatrix} \tag{10}$$

Both ε and ω present components only on the axis of rotation of the cylinder. The notation $\tilde{\omega}$ is for the antisymetric matrix attached to the vector of angular velocity :

$$\tilde{\omega} = \begin{bmatrix} 0 & -\omega_z & 0 \\ \omega_z & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{11}$$

The right member of the equation (4) is represented only by the moment of the friction force T with respect to the centre of mass G :

$$\sum \boldsymbol{M}_G = \overline{GC} \times \boldsymbol{T} \tag{12}$$

The above relation is written in matrix form as it follows:

$$\begin{bmatrix} 0 & 0 & r \\ 0 & 0 & 0 \\ -r & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -rT \end{bmatrix} \tag{12a}$$

In order to decide whether the friction regime is rolling or sliding, the calculus of the velocity of the contact point is necessary [4].

$$\boldsymbol{v}_C = \boldsymbol{v}_G + \omega \times \overline{GC} \tag{13}$$

The relation is next written in matrix form:

$$\begin{bmatrix} v_{Cx} \\ v_{Cy} \\ v_{Cz} \end{bmatrix} = \begin{bmatrix} v_{Gx} \\ v_{Gy} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -\omega_z & 0 \\ \omega_z & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ r \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} x_G \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -\dot{\theta} & 0 \\ \dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ r \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{x}_G - r\dot{\theta} \\ 0 \\ 0 \end{bmatrix} \tag{14}$$

where $\dot{x}_G$ is the velocity of the centre of mass and $\dot{\theta}$ is the angular velocity. With the aim of defining the friction force, the function *signum* denoted *sgn(x)* is useful:

$$sgn(x) = \begin{cases} 1, & if\ x > 0 \\ -1, if & x < 0 \\ 0, if & x = 0 \end{cases} \tag{15}$$

2. Finding the motion in the presence of sliding

Two distinct situations were considered:

2.1 Sliding case, $v_C \neq 0$

In this case, the relative velocity from the contact point is not zero and the system has 2 degrees of freedom (2DOF), θ and x_G and the relationship established by Amontons and Coulomb [5] exists between the normal reaction and the friction force:

$$T = -\mu N\, sgn(\dot{x}_G - r\dot{\theta}) \tag{16}$$

where μ is the coefficient of dynamic friction. Using the theorems expressed by the equations (3) and (4), the following system of equations is obtained:

$$\begin{cases} M\ddot{x}_G = Mg \sin\alpha - \mu N\, sgn(\dot{x}_G - r\dot{\theta}) \\ 0 = -Mg\cos\alpha + N \\ J_{Gz}\ddot{\theta} = \mu r N\, sgn(\dot{x}_G - r\dot{\theta}) \end{cases} \tag{17}$$

From the second equation of the system (17), the magnitude of normal reaction results:

$$N = Mg\cos\alpha \tag{18}$$

By replacing the value of N equation (18) into the other equations, a system of nonlinear differential equations is obtained:

$$\begin{cases} \ddot{x}_G = \sin\alpha - \mu\cos\alpha\, sgn(\dot{x}_G - r\dot{\theta}) \\ \ddot{\theta} = \dfrac{\mu Mrg\cos\alpha}{J_{Gz}} sgn(\dot{x}_G - r\dot{\theta}) \end{cases} \tag{19}$$

By denoting:

$$\begin{aligned} v_G &= \dot{x}_G \\ \omega &= \dot{\theta} \end{aligned} \qquad (20)$$

The system (19) and the equations (20) can be re-written under the form of a system of four differential nonlinear equations, that can be integrated numerically:

$$\frac{d}{dt}\begin{bmatrix} x_G \\ \theta \\ v_G \\ \omega \end{bmatrix} = \begin{bmatrix} v_G \\ \omega \\ g(\sin\alpha - \mu\cos\alpha)sgn(v_G - r\omega) \\ \frac{\mu Mrg\cos\alpha}{J_{Gz}} sgn(v_G - r\omega) \end{bmatrix} \qquad (20a)$$

The initial conditions written in matrix form are:

$$\begin{bmatrix} x_G \\ \theta \\ v_G \\ \omega \end{bmatrix}_{t=0} = \begin{bmatrix} x_{G0} \\ \theta_0 \\ v_{G0} \\ \omega_0 \end{bmatrix} \qquad (21)$$

where x_{G0}, v_{G0}, represent the position and velocity of the centre of mass, and θ_0, ω_0 are the angle of rotation and angular velocity, respectively, at the initial moment $t = 0$.

2.2 Pure rolling

In this case:

$$v_r = 0 \qquad (22)$$

and a relation exists between the two parameters, as the system has one degree of freedom (1DOF). The system obtained from the relations (3) and (4) has as unknowns one of the two parameters, x_G or θ, and the magnitude of the normal force N and friction force T which now are independent. The friction force in this case is written as:

$$\boldsymbol{T} = \begin{bmatrix} -T \\ 0 \\ 0 \end{bmatrix} \qquad (23)$$

The derivative of equation (14) with respect to time is effectuated, for $v_r = 0$ and a relation between the acceleration of the centre of mass $\ddot{x}_G$ and the angular acceleration $\ddot{\theta}$ is obtained:

$$\ddot{x}_G - r\ddot{\theta} = 0 \qquad (24)$$

From the two theorems the next system results:

$$\begin{cases} M\ddot{x}_G = Mg\sin\alpha + T \\ 0 = -Mg\cos\alpha + N \\ J_{Gz}\ddot{\theta} = rT \end{cases} \qquad (25)$$

with the solutions:

$$N = Mg\cos\alpha \qquad (26)$$

$$T = \frac{Mg\sin\alpha}{1 + \frac{Mr^2}{J_{Gz}}}, \qquad (27)$$

$$\ddot{\theta} = \frac{Mrg\sin\alpha}{J_{Gz} + Mr^2} \qquad (28)$$

The relations (26) and (27) show that both the normal force and the friction force have a constant value with respect to time. By integrating the equation (28), applying the initial conditions (21), the rotation angle θ is obtained:

$$\theta(t) = \theta_0 + \omega_0 t + \frac{1}{2}\frac{rMg\sin\alpha}{J_{Gz} + Mr^2}t^2 \qquad (29)$$

and also the angular velocity:

$$\omega(t) = \omega_0 + \frac{rMg\sin\alpha}{J_{Gz} + Mr^2}t \qquad (30)$$

The relations (26) and (27) permit finding the relation for the ratio T / N :

$$\frac{T}{N}=\frac{tan\alpha}{1+\frac{Mr^2}{J_{Gz}}} \qquad (31)$$

which is necessary for the validation of pure rolling condition:

$$\left|\frac{T}{N}\right|<\mu \qquad (32)$$

3. Results and discussions

The analysis of relation (31) shows that for stipulated values of the dimensions r and h of the cylinder, of its mass M and for a précised value of the angle of inclination of the plane, there is a critical value of the coefficient of friction which separates the two motion regimes, sliding or rolling.

For a tilting angle of the plane $\alpha = 30^o$, the radius of the cylinder $r = 0.12m$, the height $h = 0.2m$ and material density $\rho = 7800\, kg/m^3$, the next values are obtained:

$$M = 70.573kg \qquad (33)$$

$$J_{Gz} = 0.508kg \cdot m^2 \qquad (34)$$

$$\mu_{cr} \frac{tan\alpha}{1+\frac{Mr^2}{J_{Gz}}} = 0.192 \qquad (35)$$

For these values and for two values of the coefficient of friction; the first $\mu = 0.3 > \mu_{cr}$ and the other $\mu = 0.15 < \mu_{cr}$, considering as intial data: $x_{G0} = 0$, $\theta_0 = 0$, $v_{G0} = 0$, $\omega_0 = 0$ and $\alpha = 30^o$.

The system of differential equations (20) was integrated, corresponding to sliding. For the case of roling, the rotation θ_r and the angular velocity ω_r were found based on the relations (29) and (30), and afterthat, the displacement of the centre of mass and its velocity were found with obvious relations:

$$x_{Gr} = r\theta_r \quad ; \quad v_{Gr} = r\omega_r$$

In Figures 2 and 4, there were represented the variations of the parameters x_G (a), θ (b), v_G (c) and ω (d) obtained for $\mu = 0.3$ and $\mu = 0.15$ respectively.

In Figures 3 and 5 there were plotted the conditions of pure rolling testing, for the same values of the coefficient of friction, $\mu = 0.3$ and $\mu = 0.15$ respectively.

Form Figure 3 it is remarked that both criteria of pure rolling are fulfilled. From Figure 2 it results an important observation: although the equations (21) were obtained under the hypothesis of sliding, if the conditions of sliding existence $v_G - r\omega = 0$ and $|T/N| < \mu$ respectively, are disobeyed, the solutions of the system (21) will correspond to pure rolling.

From Figure 5 it is noticed that none of the two criteria corresponding to pure rolling is satisfied, $v - r\omega > 0$ in Figure 5a, or $T/N > \mu$, in Figure 5b.

The result is reflected in Figure 4, by the different curves corresponding to the two situations.

4. Conclusions

In the first part of the paper there were presented the components of the friction torsor from a point contact, revealing the possibility that the relative motion is made either with sliding or with rolling, and the criteria for the decision upon the two possibilities of relative motion were established. These are used in the second part of the paper, where the dynamic model in motion on an inclined plane is resumed.

First, the case of sliding motion is considered, when the system presents two degrees of freedom (2DOF) the differential equations of motion are deduced.

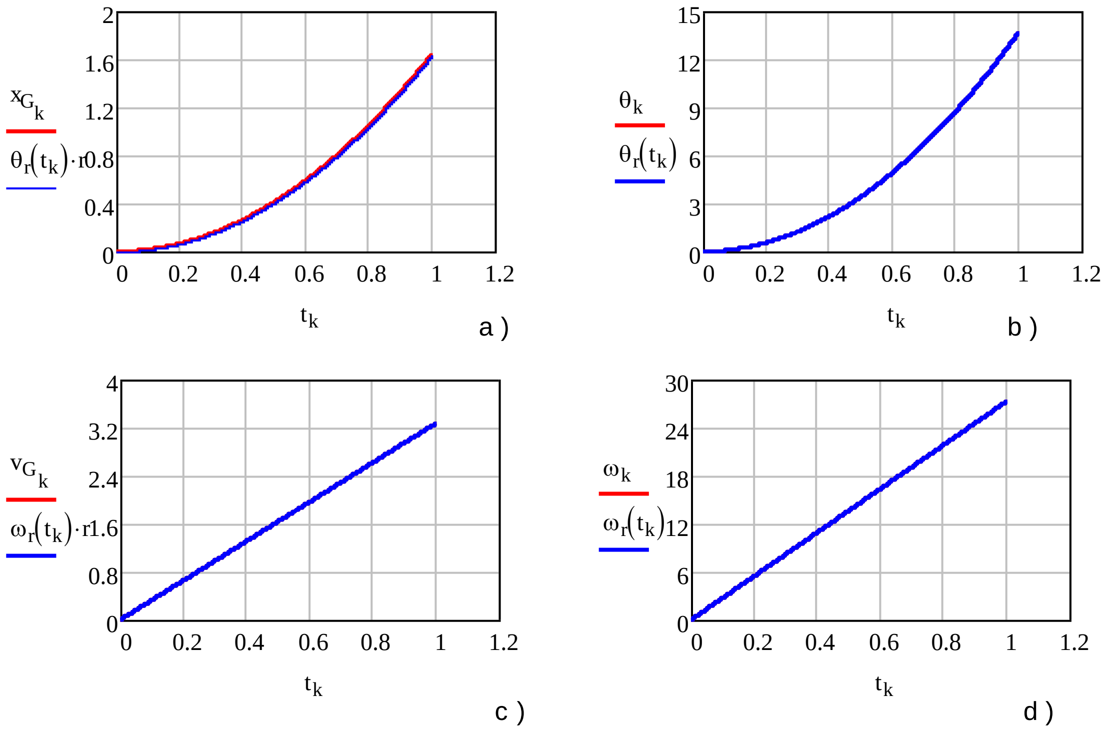


Figure 2. The variation of kinematic parameters $x_G(t)$, $\theta(t)$, $v_G(t)$ and $\omega_0(t)$, for $\alpha = 30^o$ and $\mu = 0.3$ and intial conditions $x_{G0} = 0$, $\theta_0 = 0$, $v_{G0} = 0$, $\omega_0 = 0$.

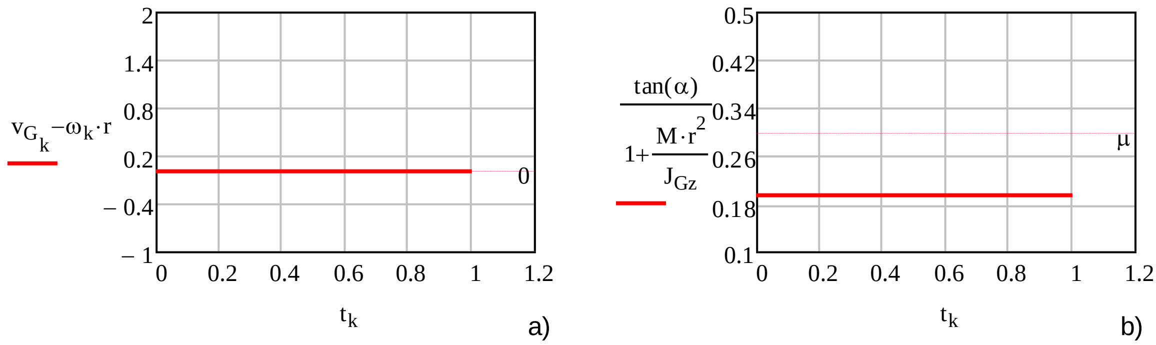
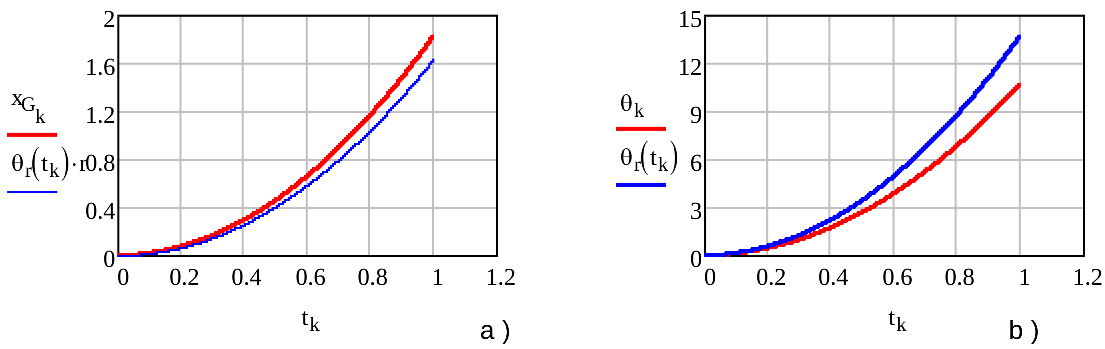


Figure 3. *Testing the kinematic* $v_G - \omega r = 0$ *and kineto-static* $|T / N| < \mu$ *criteria, for pure rolling, for the parameters* $\alpha = 30^o$ *and* $\mu = 0.3$ *and initial conditions* $x_{G0} = 0$, $\theta_0 = 0$, $v_{G0} = 0$ *and* $\omega_0 = 0$



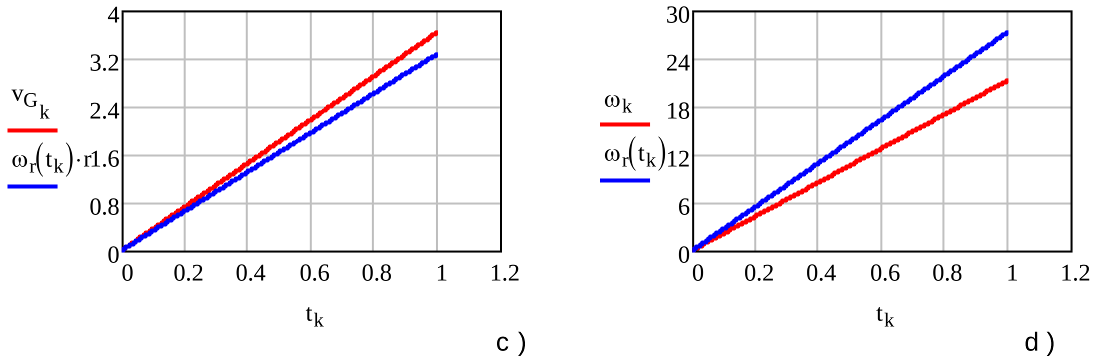


Figure 4. *The variation of kinematic parameters* $x_G(t)$, $\theta(t)$, $v_G(t)$ *and* $\omega_0(t)$ *for* $\alpha = 30^o$ *and* $\mu = 0.15$ *and intial conditions* $x_{G0} = 0$, $\theta_0 = 0$, $v_{G0} = 0$, $\omega_0 = 0$.

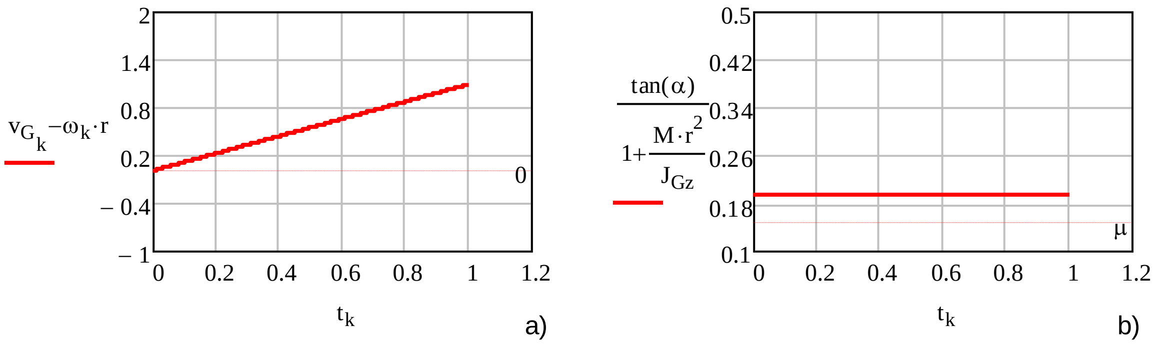


Figure 5. *testing the pure rolling kinematic criteria* $v_G - \omega r = 0$ *and the kineto-static criteria* $|T/N| < \mu$ *for* $\alpha = 30^o$; $\mu = 0.15$ *and initial conditions* $x_{G0} = 0$, $\theta_0 = 0$, $v_{G0} = 0$, $\omega_0 = 0$.

The equations of the system are non-linear and they were integrated using the Runge-Kutta 4 with constant step algorithm. Next, the equations of motion of the cylinder under the hypothesis of pure rolling are deduced. It is remarked the existence of a constant critical value of the coefficient of friction which delimitates the two domains.

An extremely important conclusion consists in the fact that when the coefficient of friction exceeds the critical value, the system performing pure rolling motion, the solutions obtained by integrating the differential equations deduced under the hypothesis of sliding are identical to the solutions obtained corresponding to the pure rolling assumptions.

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